Numerical Simulation

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Exercise Sheet 4

Closing date **May 10, 2016**.

Exercise 1. a) Prove that any strongly convergent sequence in a normed linear space is also weakly convergent. Is the opposite true?

b) Prove that in a Hilbert space $\{H, (\cdot,\cdot)\}$ the following property holds:

$$u_n \rightharpoonup u \text{ and } v_n \to v \quad \Rightarrow (u_n, v_n) \to (u, v), \quad \text{for } n \to \infty.$$

(10 points)

Exercise 2. a) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $u_a, u_b \in L^\infty(\Omega)$ with $u_a(x) \leq u_b(x)$ a.e. be given. Prove that then the set

$$\{u \in L^2(\Omega) \colon u_a(x) \leq u(x) \leq u_b(x) \quad \text{a.e. in } \Omega\}$$

is convex and closed.

b) Let $\{Y, \|\cdot\|_Y\}$ and $\{U, \|\cdot\|_U\}$ be Hilbert spaces, $y_d \in Y$, $\lambda > 0$, and a linear and continuous operator $S \colon U \to Y$ be given. Prove the strict convexity of

$$f(x) = \|Su - y_d\|_Y^2 + \lambda \|u\|_U^2$$

if $\lambda > 0$ or S is injective.

(10 points)

Exercise 3. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.
Consider the boundary value problem

$$\begin{cases} -\Delta y + uy &= f \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega \end{cases}$$

with $u, f \in L^2(\Omega)$ and $u \geq 0$ a.e. in Ω.

a) Prove existence of a solution $y \in H_0^1(\Omega)$ as well as the a priori estimate

$$\|\nabla y\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}$$

with a constant $c > 0$ independent of f and u.

b) Let now Ω be a convex polyhedron and $u \in L^3(\Omega)$. Prove that then the solution y of the boundary value problem fulfills additionally $y \in H^2(\Omega)$. Prove a corresponding estimate for $\|y\|_{H^2(\Omega)}$.

(10 points)