Numerical Simulation

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Exercise Sheet 7

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Exercise 1. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, $u, y_\Omega \in L^2(\Omega)$ be given functions, and $\lambda > 0$ be a positive real number. Moreover, $S\colon L^2(\Omega) \to L^2(\Omega)$ denotes the control-to-state operator associated with the following weak formulation

$$\text{Find } y \in V\colon \quad (\nabla y, \nabla \varphi)_{L^2(\Omega)} + (y, \varphi)_{L^2(\Omega)} = (u, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in V,$$

with $V := H_0^1(\Omega)$. Now, consider a shape regular and quasi-uniform triangulation of Ω constituting a nonoverlapping cover of Ω, which we denote by $\mathcal{T}_h = \{T\}$. Associated with this triangulation, we introduce the discrete state space

$$V \supset V_h = \{v_h \in C_0(\bar{\Omega})\colon \ v_{h_{|T}} \in \mathcal{P}^1(T) \text{ for } T \in \mathcal{T}_h\},$$

where $\mathcal{P}^1(T)$ denotes the space of polynomials of degree up to order one on each triangle T. Let $S_h\colon L^2(\Omega) \to V_h$ denote the discrete control-to-state operator associated with the discretized state equation

$$\text{Find } y_h \in V_h\colon \quad (\nabla y_h, \nabla \varphi_h)_{L^2(\Omega)} + (y_h, \varphi_h)_{L^2(\Omega)} = (u, \varphi_h)_{L^2(\Omega)} \quad \forall \varphi_h \in V_h.$$

Last, let us define continuous and discrete objective functions $f, f_h\colon L^2(\Omega) \to \mathbb{R}$, respectively, by

$$f(u) = \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}$$

and

$$f_h(u) = \frac{1}{2}\|S_h u - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}.$$

Prove the following properties:

a) For any $u_1, u_2, v \in L^2(\Omega)$, $\|f_h'(u_1)v - f_h'(u_2)v\| \leq c\|u_1 - u_2\|_{L^2(\Omega)\|v\|_{L^2(\Omega)}}$ holds with a constant $c > 0$ independent of h.

b) For any $u, v \in L^2(\Omega)$, $\|f'(u)v - f_h'(u)v\| \leq ch^2(\|u\|_{L^2(\Omega)} + \|y_\Omega\|_{L^2(\Omega)})\|v\|_{L^2(\Omega)}$ holds with a constant $c > 0$ independent of h.

c) For any $u, v \in L^2(\Omega)$, $f_h''(u)v^2 \geq c\|v\|^2$ holds with a constant $c > 0$ independet of h.

(15 points)