Scientific Computing 2

Summer term 2017
Prof. Dr. Ira Neitzel
Christopher Kacwin

universitätbonn

Sheet 0

Submission on -.

Exercise 1. (minimization/maximization problems)

Consider $f\colon \mathbb{R}^n \to \mathbb{R}$ and $X \subset \mathbb{R}^n$ nonempty. Show that the set of solutions to

$$\max_{x\in X} f(x)$$

and

$$\min_{x\in X} (-f(x))$$

are identical.

(0 points)

Exercise 2. (sublevel sets)

Consider the functions

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^2 - y^2.$$

and

$$g : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^2 + y^2.$$

a) Draw the level sets of f and g. Do they attain their minima/maxima on $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$? Add them to your drawings.

b) Show that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with $\lim_{\|x\|\to\infty} f(x) = \infty$ has compact sublevel sets $\mathcal{N}_f(w) = \{x \in \mathbb{R}^n | f(x) \leq w\}$.

c) Show that convex functions $f\colon \mathbb{R}^n \to \mathbb{R}$ have convex sublevel sets. Does the converse also hold true?

(0 points)

Exercise 3. (linear regression)

a) Let $A \in \mathbb{R}^{m,n}$ with $m > n$ and rang $A = n$, $b \in \mathbb{R}^m$ and consider

$$f : \mathbb{R}^n \mapsto \mathbb{R}, \quad f(x) = \frac{1}{2}\|Ax - b\|_2^2. \tag{1}$$

Assume that the QR-decomposition $A = QR$ is known, where $Q \in \mathbb{R}^{m,n}$, $Q^TQ = I_n$, and $R \in \mathbb{R}^{n,n}$ is an upper triangular matrix.

Calculate an optimal solution to

$$\min_{x\in\mathbb{R}^2} f(x).$$

b) Calculate the gradient and Hessian of f. Show that the Hessian is positive semi-definite, and further positive definite iff A is injective.

c) Formulate the linear regression problem in the form (1). show that for $m \in \mathbb{N}$ the matrix

$$H = 2 \begin{pmatrix} \sum_{i=1}^{m} \xi_i^2 & \sum_{i=1}^{m} \xi_i \\ \sum_{i=1}^{m} \xi_i & m \end{pmatrix}$$

is positive definite if at least two of the ξ_i are different. Discuss the relation of A in (1) and H. Solve the linear regression model using the measurements:

ξ_i	-5	-1	0	1	5
η_i	1	4	5	6	9

(0 points)

Exercise 4. (a test case)

Consider the following problem:
Find a point $x \in \mathbb{R}^2$ such that it minimizes the sum of distances to three given points $x_1, x_2, x_3 \in \mathbb{R}^2$.

a) Formulate this problem as an optimization problem and show that a solution x^* exists. Is it unique?

b) Let $x^* \neq x_i$, $i = 1 \ldots 3$. Characterize x^* with the first order necessary condition for optimality and geometrical considerations.

(0 points)